

# Linear programming: examples

## Advanced Automation and Control

Dipartimento di Ingegneria Industriale e dell'Informazione  
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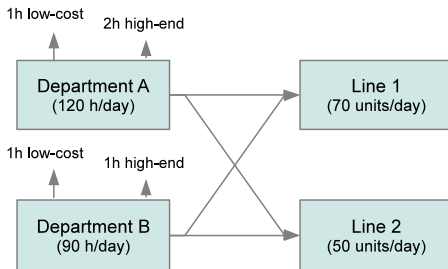
Slides adapted from the material of Prof. Ferrari Trecate (former lecturer of this course)

# LP and management science

## Typical decision problems in industry

- Product mix
- Diet problems
- Blending problems
- Transport problems
- Product mix with resource allocation
- Multiperiod production planning
- Portfolio optimization
- ...

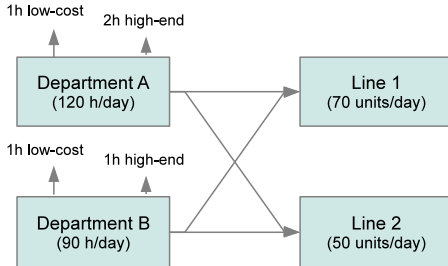
# Product mix



A company manufactures two radio models (low-cost and high-end) and produces two components

- Department A: antennas (max. 120h hours of production per day)
  - ▶ 1h of work for a low-cost antenna
  - ▶ 2h of work for a high-end antenna
- Department B: case (max. 90h hours of production per day)
  - ▶ 1h of work for a low-cost case
  - ▶ 1h of work for a high-end case

# Product mix



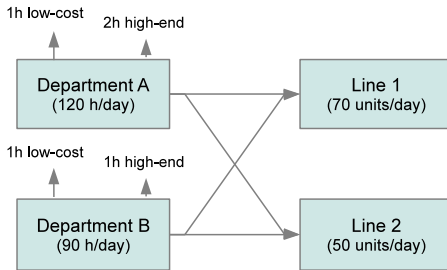
The company has two assembly lines (1 radio=1 antenna + 1 case)

- Line 1: production of low-cost models. No more than 70 units/day
- Line 2: production of high-end models. No more than 50 units/day

Profits: 20 Euros for a low-cost radio and 30 Euros for a high-end radio.

Assuming the company will sell all radios, which is the optimal number of units, for each model, that must be produced daily for maximizing the revenue?

# Product mix

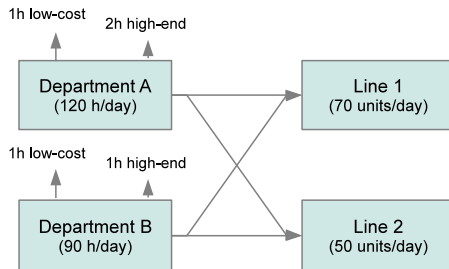


## Choice of variables

Usually this is the most difficult step in representing decision problems as optimization problems !

Guideline: cost and constraints must be a function of optimization variables only.

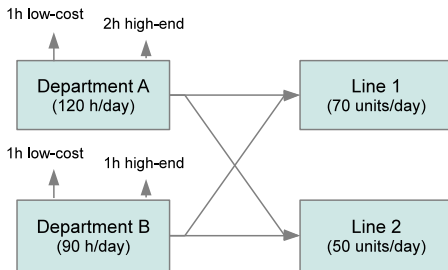
# Product mix



## Choice of variables - product mix

- $x_1$ : number of produced low-cost radios per day
- $x_2$ : number of produced high-end radios per day

# Product mix



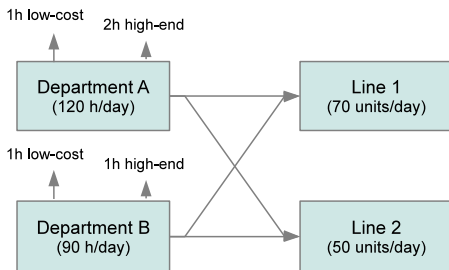
## Choice of variables - product mix

- $x_1$ : number of produced low-cost radios per day
- $x_2$ : number of produced high-end radios per day

## Cost and type of problem

Cost:  $20x_1 + 30x_2$ , to be *maximized*

# Product mix



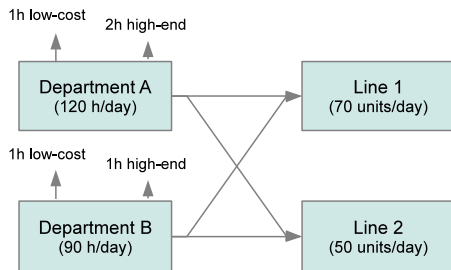
## Constraints

Capacity constraints of assembly lines

- $x_1 \leq 70$  (line 1)
- $x_2 \leq 50$  (line 2)



# Product mix



## Constraints

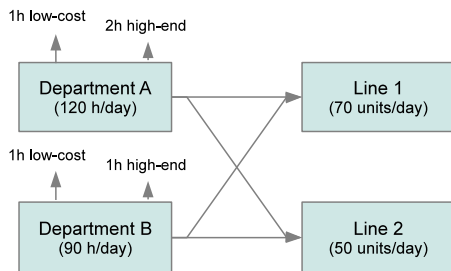
Capacity constraints of assembly lines

- $x_1 \leq 70$  (line 1)
- $x_2 \leq 50$  (line 2)

Capacity constraints of production departments

- $x_1 + 2x_2 \leq 120$  (department 1)
- $x_1 + x_2 \leq 90$  (department 2)

# Product mix



## Constraints

Capacity constraints of assembly lines

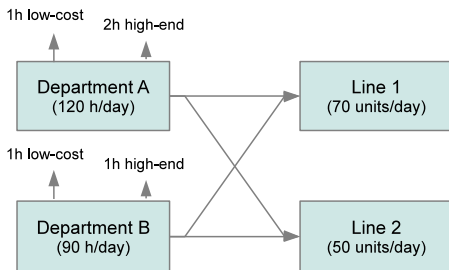
- $x_1 \leq 70$  (line 1)
- $x_2 \leq 50$  (line 2)

Capacity constraints of production departments

- $x_1 + 2x_2 \leq 120$  (department 1)
- $x_1 + x_2 \leq 90$  (department 2)

Positivity constraints:  $x_1 \geq 0$ ,  $x_2 \geq 0$

# Product mix



## LP problem

$$\begin{aligned} \max_{x_1, x_2} \quad & 20x_1 + 30x_2 \\ & x_1 \leq 70 \\ & x_2 \leq 50 \\ & x_1 + 2x_2 \leq 120 \\ & x_1 + x_2 \leq 90 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

# Product mix

## LP problem

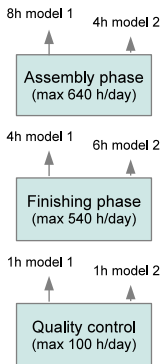
$$\begin{aligned} \max_{x_1, x_2} \quad & 20x_1 + 30x_2 \\ x_1 \quad & \leq 70 \\ x_2 \quad & \leq 50 \\ x_1 + 2x_2 \quad & \leq 120 \\ x_1 + x_2 \quad & \leq 90 \\ x_1 \quad & \geq 0 \\ x_2 \quad & \geq 0 \end{aligned}$$

## LP problem - matrix notation

$$\begin{aligned} \max \quad & c^T x \\ Ax \leq & b \\ x \geq & 0 \end{aligned}$$

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 20 \\ 30 \end{bmatrix} \\ A &= \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 120 \\ 90 \\ 70 \\ 50 \end{bmatrix} \end{aligned}$$

# Product mix revised

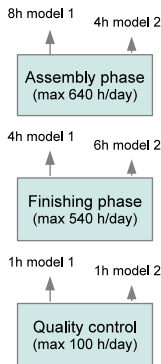


A company manufactures two models of cases for telephones (model 1 and model 2). The production cycle comprises three phases with bounded resources that must be allocated to the two products

- Each phase is modeled through the maximal availability of men-hours per day and men-hours required to process a single unit
- Profits: 30 Euros for a model 1 unit and 20 Euros for a model 2 unit

Assuming all cases will be sold, which is the optimal number of units, for each model, that must be produced daily for maximizing the revenue?

# Product mix revised



## Choice of variables

- $x_1$ : n. of model 1 units
- $x_2$ : n. of model 2 units

## Cost and type of problem

Cost:  $30x_1 + 20x_2$ , to be  
*maximized*

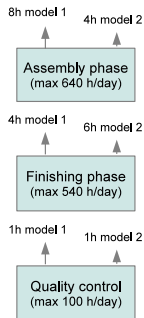
## Constraints

Capacity constraints in each phase

- $8x_1 + 4x_2 \leq 640$  (assembly)
- $4x_1 + 6x_2 \leq 540$  (finishing)
- $x_1 + x_2 \leq 100$  (quality control)

Positivity constraints:  $x_1 \geq 0$ ,  $x_2 \geq 0$

# Product mix revised



## LP problem

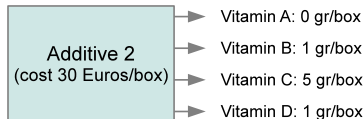
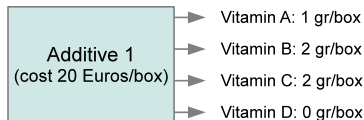
$$\begin{aligned} \max_{x_1, x_2} \quad & 30x_1 + 20x_2 \\ 8x_1 + 4x_2 \quad & \leq 640 \\ 4x_1 + 6x_2 \quad & \leq 540 \\ x_1 + x_2 \quad & \leq 100 \\ x_1 \quad & \geq 0 \\ x_2 \quad & \geq 0 \end{aligned}$$

## LP problem - matrix notation

$$\begin{aligned} \max \quad & c^T x \\ Ax \leq & b \\ x \geq & 0 \end{aligned}$$

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 30 \\ 20 \end{bmatrix} \\ A &= \begin{bmatrix} 8 & 4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 640 \\ 540 \\ 100 \end{bmatrix} \end{aligned}$$

# Diet problem



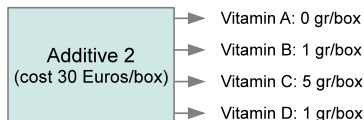
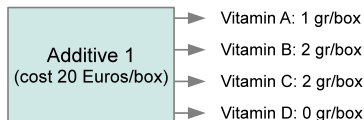
A company producing chicken food can use two types of additives with different vitamin contents. The product is obtained by mixing additives in a suitable quantity

- Vitamin contents and additive costs are given in the picture
- 1 box of chicken food must contain at least 2 gr. of vitamin A, at least 12 gr. of vitamin B, at least 36 gr. of vitamin C and at least 4 gr. of vitamin D

Which is the quantity of additives that allows the company to produce one box of chicken food while minimizing the costs ?



# Diet problem



## Choice of variables

- $x_1$ : n. of boxes of Additive 1
- $x_2$ : n. of boxes of Additive 2

## Cost and type of problem

Cost:  $20x_1 + 30x_2$ , to be *minimized*

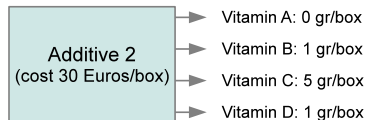
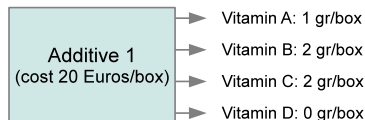
## Constraints

Minimal quantities of vitamins

- $x_1 \geq 2$  (vitamin A)
- $2x_1 + x_2 \geq 12$  (vitamin B)
- $2x_1 + 5x_2 \geq 36$  (vitamin C)
- $x_2 \geq 4$  (vitamin D)

Positivity constraints:  $x_1 \geq 0, x_2 \geq 0$

# Diet problem



## LP problem

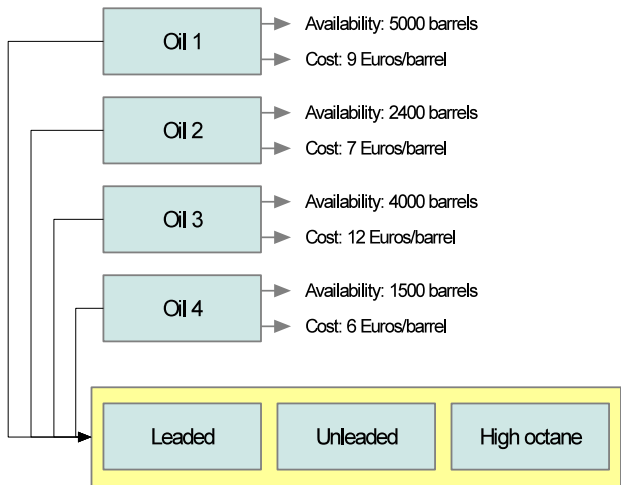
$$\begin{aligned} \min_{x_1, x_2} \quad & 20x_1 + 30x_2 \\ & x_1 \geq 2 \\ & 2x_1 + x_2 \geq 12 \\ & 2x_1 + 5x_2 \geq 36 \\ & x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

## LP problem - matrix notation

$$\begin{aligned} \min \quad & c^T x \\ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad c = \begin{bmatrix} 20 \\ 30 \end{bmatrix} \\ A &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 5 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 12 \\ 36 \\ 4 \end{bmatrix} \end{aligned}$$

## Blending problem



A company produces 3 types of gasoline (leaded, unleaded, high-octane) blending 4 types of oils in suitable proportions

# Blending problem

## Problem data

- The maximal number of available barrels and the buying costs are given in the picture
- Each gasoline type must fulfill the following composition requirements

GASOLINE	% of Oil 1	% of Oil 2	% of Oil 3	% of Oil 4
Leaded	at least 40%	no more than 20%	at least 30%	
Unleaded			at least 40%	
High octane	at least 10%	no more than 50%		

- Gasoline selling prices are: 12 Euros/barrel for leaded, 18 Euros/barrel for unleaded and 10 Euros/barrel for high-octane

Which are the optimal quantities of the 4 components that have to be bought in order to maximize the profits (profits = revenues - costs) ?

# Blending problem

## Choice of variables

$x_{ij}$ : n. of barrels of oil  $i$  used for gasoline of type  $j$ ,  $i = 1, 2, 3, 4$ ,  
 $j \in \{L, U, H\}$ ,  $L =$  Leaded,  $U =$  Unleaded,  $H =$  High-octane

## Cost and type of problem

Cost to be *maximized*

$$\underbrace{12 \sum_{i=1}^4 x_{iL} + 18 \sum_{i=1}^4 x_{iU} + 10 \sum_{i=1}^4 x_{iH}}_{\text{revenues}} - \underbrace{9 \sum_{j \in \{U, L, H\}} x_{1j} - 7 \sum_{j \in \{U, L, H\}} x_{2j} - 12 \sum_{j \in \{U, L, H\}} x_{3j} - 6 \sum_{j \in \{U, L, H\}} x_{4j}}_{\text{costs}}$$

# Blending problem

## Constraints

Maximal availability of barrels

- $x_{1L} + x_{1U} + x_{1H} \leq 5000$  (Oil 1)
- $x_{2L} + x_{2U} + x_{2H} \leq 2400$  (Oil 2)
- $x_{3L} + x_{3U} + x_{3H} \leq 4000$  (Oil 3)
- $x_{4L} + x_{4U} + x_{4H} \leq 1500$  (Oil 4)

Composition requirements

$\frac{x_{1L}}{x_{1L} + x_{2L} + x_{3L} + x_{4L}} \geq 0.4$	in affine form	$0.6x_{1L} - 0.4x_{2L} - 0.4x_{3L} - 0.4x_{4L} \geq 0$
$\frac{x_{2L}}{x_{1L} + x_{2L} + x_{3L} + x_{4L}} \leq 0.2$	in affine form	$-0.2x_{1L} + 0.8x_{2L} - 0.2x_{3L} - 0.2x_{4L} \leq 0$
$\frac{x_{3L}}{x_{1L} + x_{2L} + x_{3L} + x_{4L}} \geq 0.3$	in affine form	$-0.3x_{1L} - 0.3x_{2L} + 0.7x_{3L} - 0.3x_{4L} \geq 0$
$\frac{x_{3U}}{x_{1U} + x_{2U} + x_{3U} + x_{4U}} \geq 0.4$	in affine form	$-0.4x_{1U} - 0.4x_{2U} + 0.6x_{3U} - 0.4x_{4U} \geq 0$
$\frac{x_{1H}}{x_{1H} + x_{2H} + x_{3H} + x_{4H}} \geq 0.1$	in affine form	$0.9x_{1H} - 0.1x_{2H} - 0.1x_{3H} - 0.1x_{4H} \geq 0$
$\frac{x_{2H}}{x_{1H} + x_{2H} + x_{3H} + x_{4H}} \leq 0.5$	in affine form	$-0.5x_{1H} + 0.5x_{2H} - 0.5x_{3H} - 0.5x_{4H} \leq 0$

Positivity constraints:  $x_{ij} \geq 0$ ,  $i = 1, 2, 3, 4$ ,  $j \in \{U, L, H\}$

# Blending problem

## LP problem - matrix notation

$$\begin{aligned} \max \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

with

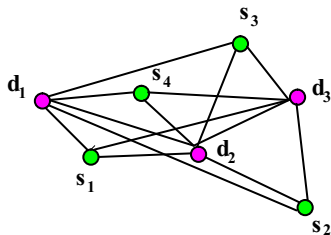
$$x = [x_{1L} \quad x_{2L} \quad x_{3L} \quad x_{4L} \quad x_{1U} \quad x_{2U} \quad x_{3U} \quad x_{4U} \quad x_{1H} \quad x_{2H} \quad x_{3H} \quad x_{4H}]^T$$

$$c = [3 \quad 5 \quad 0 \quad 6 \quad 9 \quad 11 \quad 6 \quad 12 \quad 1 \quad 3 \quad -2 \quad 4]^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ -0.6 & 0.4 & 0.4 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0.8 & -0.2 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.3 & -0.7 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0.4 & -0.6 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.9 & 0.1 & 0.1 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0.5 & -0.5 & -0.5 \end{bmatrix}$$

$$b = [5000 \quad 2400 \quad 4000 \quad 1500 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$$

# Transport problem



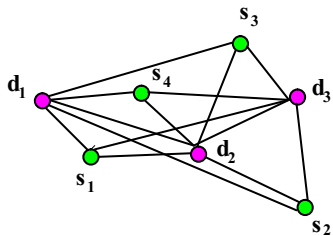
A company owns  $n$  fuel storage points (i.e.  $n$  supply points) and it must supply  $m$  gas stations (i.e.  $m$  demand points). Every gas station can be reached from all storage points.

## Data and requirements

- Every storage point has a maximal availability of  $a_i \geq 0$   $i = 1, \dots, n$  liters of fuel
- Every gas station specifies a demand of  $b_j \geq 0$ ,  $j = 1, \dots, m$  liters of fuel
- $t_{ij} \geq 0$  is the cost for shipping 1 liter of fuel from storage point  $i$  to gas station  $j$



# Transport problem



Which are the optimal quantities of fuel to be shipped from each storage point to each gas station in order to minimize transport costs while fulfilling exactly the demands ?

# Transport problem

## Choice of variables

$x_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , n. of liters of fuel shipped from storage point  $i$  to gas station  $j$

## Cost and type of problem

Cost to be *minimized*

$$\sum_{i=1}^n \sum_{j=1}^m t_{ij} x_{ij}$$

## Constraints

Maximal availability of fuel at storage points

- $\sum_{j=1}^m x_{ij} \leq a_i$ ,  $i = 1, \dots, n$

Demand constraints

- $\sum_{i=1}^n x_{ij} = b_j$ ,  $j = 1, \dots, m$

Positivity constraints:  $x_{ij} \geq 0$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$

# Transport problem

## LP problem

$$\min \sum_{i=1}^n \sum_{j=1}^m t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^m x_{ij} &\leq a_i, \quad i = 1, \dots, n \\ \sum_{i=1}^n x_{ij} &= b_j, \quad j = 1, \dots, m \\ x_{ij} &\geq 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m \end{aligned} \quad (\mathbf{A})$$

## Remarks

- Constraints (A) can be replaced by  $\sum_{i=1}^n x_{ij} \geq b_j, j = 1, \dots, m$ . Why ?

# Transport problem

## LP problem

$$\min \sum_{i=1}^n \sum_{j=1}^m t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^m x_{ij} &\leq a_i, \quad i = 1, \dots, n \\ \sum_{i=1}^n x_{ij} &= b_j, \quad j = 1, \dots, m \\ x_{ij} &\geq 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m \end{aligned} \quad (\mathbf{A})$$

## Remarks

- Constraints (A) can be replaced by  $\sum_{i=1}^n x_{ij} \geq b_j, j = 1, \dots, m$ . Why ?
- The LP problem is always feasible ?

# Transport problem

## LP problem

$$\min \sum_{i=1}^n \sum_{j=1}^m t_{ij} x_{ij}$$

$$\begin{aligned} \sum_{j=1}^m x_{ij} &\leq a_i, \quad i = 1, \dots, n \\ \sum_{i=1}^n x_{ij} &= b_j, \quad j = 1, \dots, m \\ x_{ij} &\geq 0, \quad i = 1, \dots, n, \quad j = 1, \dots, m \end{aligned} \quad (\mathbf{A})$$

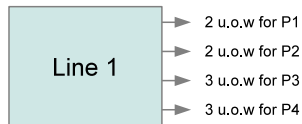
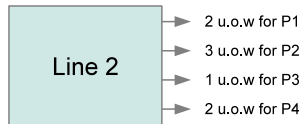
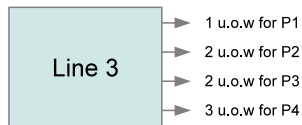
## Remarks

- Constraints (A) can be replaced by  $\sum_{i=1}^n x_{ij} \geq b_j, j = 1, \dots, m$ . Why ?
- The LP problem is always feasible ? **NO !**
  - ▶ A necessary and sufficient condition for feasibility is

$$\sum_{i=1}^n a_i \geq \sum_{j=1}^m b_j$$

# Product mix with resource allocation

In standard product mix problems, capacity constraints are fixed. We now assume to have resources that can be allocated to different production phases



A food company produces 4 products  $P_i$ ,  $i = 1, 2, 3, 4$ . Each product must go through 3 production lines  $L_j$ ,  $j = 1, 2, 3$ . Each line is modeled through the units of work (u.o.w.) required to process a single unit of product

# Product mix with resource allocation

## Requirements

- At least 1000 units of  $P_2$  and no more than 500 units of  $P_1$  must be produced
- U.o.w are men-hours that fall in 7 categories  $T_i$ ,  $i = 1, \dots, 7$  according to the versatility of workers

Category	Destination line	Max. availability
$T_1$	Line 1	12000
$T_2$	Line 2	7000
$T_3$	Line 3	9000
$T_4$	Lines 1 and 2	4000
$T_5$	Lines 1 and 3	3000
$T_6$	Lines 2 and 3	3000
$T_7$	Lines 1,2 and 3	2000

- Profits per unit are 10 Euros for  $P_1$ , 12 Euros for  $P_2$ , 13 Euros for  $P_3$  and 14 Euros for  $P_4$

Assuming all products will be sold, which is a joint production and resource allocation plan that maximizes the profits ?

# Product mix with resource allocation

## Choice of variables

Two categories of variables are needed

- $P_i, i = 1, \dots, 4$ : units of product  $i$
- $T_4L_1, T_4L_2, T_5L_1, T_5L_3, T_6L_2, T_6L_3, T_7L_1, T_7L_2, T_7L_3$ . For instance  $T_4L_1$  is the number of men-hours of category  $T_4$  allocated to line 1.

## Cost and type of problem

Cost to be *maximized*

$$10P_1 + 12P_2 + 13P_3 + 14P_4$$



# Product mix with resource allocation

## Constraints

Capacity constraints of production lines

- $2P_1 + 2P_2 + 3P_3 + 3P_4 \leq 12000 + T_4L_1 + T_5L_1 + T_7L_1$  (line 1)
- $2P_1 + 3P_2 + P_3 + 2P_4 \leq 7000 + T_4L_2 + T_6L_2 + T_7L_2$  (line 2)
- $P_1 + 2P_2 + 2P_3 + 3P_4 \leq 9000 + T_5L_3 + T_6L_3 + T_7L_3$  (line 3)

Bounds on the number of products

- $P_2 \geq 1000$
- $P_4 \leq 500$

Maximal availability of u.o.w. for versatile workers

- $T_4L_1 + T_4L_2 \leq 4000$
- $T_5L_1 + T_5L_3 \leq 3000$
- $T_6L_2 + T_6L_3 \leq 3000$
- $T_7L_1 + T_7L_2 + T_7L_3 \leq 2000$

Positivity constraints:

$$P_1, P_2, P_3, P_4, T_4L_1, T_4L_2, T_5L_1, T_5L_3, T_6L_2, T_6L_3, T_7L_1, T_7L_2, T_7L_3 \geq 0$$

# Product mix with resource allocation

## LP problem

$$\max 10P_1 + 12P_2 + 13P_3 + 14P_4$$

$$2P_1 + 2P_2 + 3P_3 + 3P_4 \leq 12000 + T_4L_1 + T_5L_1 + T_7L_1$$

$$2P_1 + 3P_2 + P_3 + 2P_4 \leq 7000 + T_4L_2 + T_6L_2 + T_7L_2$$

$$P_1 + 2P_2 + 2P_3 + 3P_4 \leq 9000 + T_5L_3 + T_6L_3 + T_7L_3$$

$$P_2 \geq 1000$$

$$P_4 \leq 500$$

$$T_4L_1 + T_4L_2 \leq 4000$$

$$T_5L_1 + T_5L_3 \leq 3000$$

$$T_6L_2 + T_6L_3 \leq 3000$$

$$T_7L_1 + T_7L_2 + T_7L_3 \leq 2000$$

$$P_1, P_2, P_3, P_4, T_4L_1, T_4L_2, T_5L_1, T_5L_3 \geq 0$$

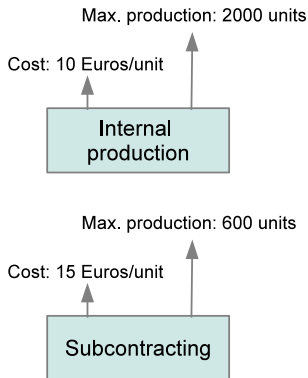
$$T_6L_2, T_6L_3, T_7L_1, T_7L_2, T_7L_3 \geq 0$$

## Exercise

Write the LP problem in matrix notation

# Multiperiod production planning

A company must determine how many washing machines should be produced during each of the next 5 months. Washing machines production can be internal or subcontracted to another company. Production capacities and costs are given in the figure



# Multiperiod production planning

## Data and Requirements

- Inventory cost: 2 Euros/unit for a whole month
- Inventory at the beginning of the first month: 300 units
- Inventory at the end of the fifth month: 300 units
- Demand during each of the next five months (the company must meet demands on time)

Month	Demand
1	1200
2	2100
3	2400
4	3000
5	4000

How many washing machines must be produced (internally and by the subcontractor), each month, in order to meet demands on time and minimize the costs ?

# Multiperiod production planning

## Choice of variables

Three categories of variables are needed

- $P_i, i = 1, \dots, 5$ : units produced internally during month  $i$
- $S_i, i = 1, \dots, 5$ : units produced by the subcontractor during month  $i$
- $I_i, i = 1, \dots, 4$ : inventory at the end of month  $i$

## Cost and type of problem

Cost to be *minimized*

$$10 \sum_{i=1}^5 P_i + 15 \sum_{i=1}^5 S_i + 2 \sum_{i=1}^4 I_i$$

# Multiperiod production planning

## Constraints

### Maximal production

- $P_i \leq 2000, i = 1, \dots, 5$  (internal production)
- $S_i \leq 600, i = 1, \dots, 5$  (subcontracting)

### Balance constraints: for month $i$ ,

Inventory at the end of month  $i =$  Inventory at the end of month  $i - 1 +$   
+ month  $i$  production  $-$  month  $i$  demand

- $I_1 = 300 + P_1 + S_1 - 1200$
- $I_2 = +I_1 + P_2 + S_2 - 2100$
- $I_3 = +I_2 + P_3 + S_3 - 2400$
- $I_4 = +I_3 + P_4 + S_4 - 3000$
- $300 = +I_4 + P_5 + S_5 - 4000$

Positivity constraints:  $P_i, S_i \geq 0, i = 1 \dots, 5, I_j \geq 0, j = 1 \dots, 4$

# Multiperiod production planning

## LP problem

$$\begin{aligned} \min \quad & 10 \sum_{i=1}^5 P_i + 15 \sum_{i=1}^5 S_i + 2 \sum_{i=1}^4 I_i \\ & P_i \leq 2000, \quad i = 1, \dots, 5 \\ & S_i \leq 600, \quad i = 1, \dots, 5 \\ & I_1 - P_1 - S_1 = 300 - 1200 \\ & I_2 - I_1 - P_2 - S_2 = -2100 \\ & I_3 - I_2 - P_3 - S_3 = -2400 \\ & I_4 - I_3 - P_4 - S_4 = -3000 \\ & -I_4 - P_4 - S_4 = -4000 - 300 \\ & P_i, S_i \geq 0, \quad i = 1, \dots, 5 \\ & I_j \geq 0, \quad j = 1, \dots, 4 \end{aligned}$$

# Portfolio optimization

A manager must decide how to invest 500000 Euros in different financial products. His goal is to maximize earnings while avoiding high risk exposure

Financial products and expected return on investment

Financial product	Market	Return %
$T_1$	Cars - Germany	10.3
$T_2$	Cars - Japan	10.1
$T_3$	Computers - USA	11.8
$T_4$	Computers - USA	11.4
$T_5$	Household appliances - Europe	12.7
$T_6$	Household appliances - Asia	12.2
$T_7$	Insurance - Germany	9.5
$T_8$	Insurance - USA	9.9
$T_9$	BOT	3.6
$T_{10}$	CCT	4.2



# Portfolio optimization

## Investment requirements

- 1 No more than 150000 Euros in the car options
- 2 No more than 150000 Euros in the computer options
- 3 No more than 100000 Euros in the appliance options
- 4 At least 100000 Euros in the insurance options
- 5 At least 125000 Euros in BOT or CCT
- 6 At least 40% of the money invested in CCT must be invested in BOT
- 7 No more than 250000 Euros must be invested in German options
- 8 No more than 200000 Euros must be invested in USA options

# Portfolio optimization

## Choice of variables

$T_i$ ,  $i = 1, \dots, 10$ : thousands of Euros invested in the financial product  $i$

## Cost and type of problem

Cost to be *maximized*

$$0.103T_1 + 0.101T_2 + 0.118T_3 + 0.114T_4 + 0.127T_5 + 0.122T_6 + 0.095T_7 + \\ + 0.099T_8 + 0.036T_9 + 0.042T_{10}$$

# Portfolio optimization

## Constraints

Total investment:  $\sum_{i=1}^{10} T_i = 500$

Investment requirements:

- 1  $T_1 + T_2 \leq 150$
- 2  $T_3 + T_4 \leq 150$
- 3  $T_5 + T_6 \leq 100$
- 4  $T_7 + T_8 \geq 100$
- 5  $T_9 + T_{10} \geq 125$
- 6  $T_9 - 0.4T_{10} \geq 0$
- 7  $T_1 + T_7 \leq 250$
- 8  $T_3 + T_4 + T_8 \leq 200$

Positivity constraints:  $T_i \geq 0, i = 1 \dots, 10$